REAL HYPERSURFACES EQUIPPED WITH PSEUDO-PARALLEL STRUCTURE JACOBI OPERATOR IN $\mathbb{C}P^2$ AND $\mathbb{C}H^2$

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ABSTRACT. Motivated by the work done in [4], [5], [12] and [15], we classify real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ equipped with pseudo-parallel structure Jacobi operator.

1 Introduction

A complex n-dimensional Kaehler manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form is complex analytically isometric to a complex projective space $\mathbb{C}P^n$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $\mathbb{C}H^n$ if $c > 0, c = 0$ or $c < 0$ respectively.

Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. Then an almost contact metric structure (φ, ξ, η, g) can be defined on M induced from the Kaehler metric and complex structure J on $M_n(c)$. The structure vector field ξ is called principal if $A\xi = \alpha\xi$, where A is the shape operator of M and $\alpha = \eta(A\xi)$ is a smooth function. A real hypersurface is said to be a *Hopf hypersurface* if ξ is principal.

The classification problem of real hypersurfaces in complex space forms is of great importance in Differential Geometry. The study of this was initiated by Takagi [18], [17], who classified all homogenous real hypersurfaces in $\mathbb{C}P^n$ into six types, which are said to be of type A_1, A_2, B, C, D and E. In [3] Hopf hypersurfaces were considered as tubes over certain submanifolds in $\mathbb{C}P^n$. In [9] the local classification theorem for Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$ was given. In the case of complex hyperbolic space $\mathbb{C}H^n$, the classification theorem for Hopf hypersurfaces with constant principal curvatures was given by Berndt [1].

Okumura [13], in $\mathbb{C}P^n$, and Montiel and Romero [10], in $\mathbb{C}H^n$, gave the classification of real hypersurfaces satisfying relation $A\varphi = \varphi A$.

Theorem 1.1 *Let M be a real hypersurface of $M_n(c)$, $n \geq 2$ $(c \neq 0)$. If it satisfies $A\varphi - \varphi A = 0$, then M is locally congruent to one of the following hypersurfaces:*

- *In case $\mathbb{C}P^n$*

  (A_1) *a geodesic hypersphere of radius r , where $0 < r < \frac{\pi}{2}$,*

  (A_2) *a tube of radius r over a totally geodesic $\mathbb{C}P^k$,($1 \leq k \leq n-2$), where $0 < r < \frac{\pi}{2}$.*

- *In case $\mathbb{C}H^n$*

  (A_0) *a horosphere in $\mathbb{C}H^n$, i.e a Montiel tube,*

  (A_1) *a geodesic hypersphere or a tube over a hyperplane $\mathbb{C}H^{n-1}$,*

  (A_2) *a tube over a totally geodesic $\mathbb{C}H^k$ ($1 \leq k \leq n-2$).*

The Jacobi operator with respect to X on M is defined by $R(\cdot, X)X$, where R is the Riemmanian curvature of M. For $X = \xi$ the Jacobi operator is called structure Jacobi operator and is denoted by $l = R(\cdot, \xi)\xi$. It has a fundamental role in almost contact manifolds. Many differential geometers have studied real hypersurfaces in terms of the structure Jacobi operator.

The study of real hypersurfaces whose structure Jacobi operator satisfies conditions concerned to the parallelness of it is a problem of great importance. In [14] the nonexistence of real hypersurfaces in nonflat complex space form with parallel structure Jacobi operator ($\nabla l = 0$) was proved. In [16] a weaker condition ($\mathbb{D}$-parallelness), that is $\nabla_X l = 0$ for any vector field X orthogonal to ξ, was studied and it was proved the nonexistence of such real hypersurfaces in case of $\mathbb{C}P^n$ ($n \geq 3$). The ξ-parallelness of structure Jacobi operator in combination with other conditions was another problem that was studied by many other authors such as Ki, Perez, Santos, Suh ([8]).

A tensor field P of type (1, s) is said to be *semi-parallel* if $R \cdot P = 0$, where R acts on P as a derivation.

More generally, it is said to be *pseudo-parallel* if there exists a function L such that

$$R \cdot P = L\{(X \wedge Y) \cdot P\},$$

where $(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y$. If $L \neq 0$, then the pseudo-parallel tensor is called *proper*.

A Riemannian manifold M is said to be *semi-symmetric* if $R \cdot R = 0$, where the Riemannian curvature tensor R acts on R as a derivation. Deszcz in [6] introduced the notion of *pseudo-symmetry*. A Riemannian manifold is said to be *pseudo-symmetric* if there exists a function L such that $R(X, Y) \cdot R = L\{(X \wedge Y) \cdot R\}$. If L is a constant then the pseudo-symmetric space is called a *pseudo-symmetric space of constant type*. Both of these notions were studied in the case of real hypersurfaces in complex space forms. More precisely, in [12] Niebergall and Ryan proved the non-existence of semi-symmetric Hopf real hypersurfaces and recently in [5] Cho, Hamada and Inoguchi gave the classification of pseudo-symmetric Hopf real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$.

Recently, in [15] Perez and Santos proved that there exist no real hypersurfaces in complex projective space $\mathbb{C}P^n$, $n \geq 3$, with semi-parallel structure Jacobi operator, (i.e.

$R \cdot l = 0$). Cho and Kimura in [4] generalized the previous work and proved the non-existence of real hypersurfaces in complex space forms, whose structure Jacobi operator is semi-parallel.

From the above raises naturally the question:

"Do there exist real hypersurfaces with pseudo-parallel structure Jacobi operator?"

In this paper, we study real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ equipped with *pseudo-parallel structure Jacobi operator*, i.e. the structure Jacobi operator satisfies the following condition:

$$R(X,Y) \cdot l = L\{(X \wedge Y) \cdot l\},$$

more precisely:

$$R(X,Y)lZ - l(R(X,Y)Z) = L\{(X \wedge Y)lZ - l((X \wedge Y)Z)\}, \qquad (1.1)$$

with $L \neq 0$.

Even though Cho and Kurihara proved in [4] the non-existence of real hypersurfaces in complex space form, whose structure Jacobi operator is semi-parallel, in the present paper we prove the existence of real hypersurfaces, whose structure Jacobi operator is pseudo-parallel and we classify them. More precisely:

Main Theorem: *Every real hypersurface M in $\mathbb{C}P^2$ or $\mathbb{C}H^2$, equipped with pseudo-parallel structure Jacobi operator is a Hopf hypersurface.*
In case of $\mathbb{C}P^2$, M is locally congruent to:

- *a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{2}$,*

- *or to a non-homogeneous real hypersurface, which is considered as a tube of radius $\frac{\pi}{4}$ over a holomorphic curve in $\mathbb{C}P^2$.*

In case of $\mathbb{C}H^2$, M is locally congruent to:

- *a horosphere,*

- *or to a geodesic hypersphere,*

- *or to a tube over $\mathbb{C}H^1$,*

- *or to a Hopf hypersurface with $\eta(A\xi) = 0$ in $\mathbb{C}H^2$.*

2   Preliminaries

Throughout this paper all manifolds, vector fields e.t.c. are assumed to be of class C^∞ and all manifolds are assumed to be connected. Furthermore, the real hypersurfaces are supposed to be oriented and without boundary. Let M be a real hypersurface immersed in a nonflat complex space form $(M_n(c), G)$ with almost complex structure J of constant

holomorphic sectional curvature c. Let N be a unit normal vector field on M and $\xi = -JN$. For a vector field X tangent to M we can write $JX = \varphi(X) + \eta(X)N$, where φX and $\eta(X)N$ are the tangential and the normal component of JX respectively. The Riemannian connection $\overline{\nabla}$ in $M_n(c)$ and ∇ in M are related for any vector fields X, Y on M:

$$\overline{\nabla}_Y X = \nabla_Y X + g(AY, X)N,$$

$$\overline{\nabla}_X N = -AX,$$

where g is the Riemannian metric on M induced from G of $M_n(c)$ and A is the shape operator of M in $M_n(c)$. M has an almost contact metric structure (φ, ξ, η) induced from J on $M_n(c)$ where φ is a (1,1) tensor field and η a 1-form on M such that ([2])

$$g(\varphi X, Y) = G(JX, Y), \qquad \eta(X) = g(X, \xi) = G(JX, N).$$

Then we have

$$\varphi^2 X = -X + \eta(X)\xi, \qquad \eta \circ \varphi = 0, \qquad \varphi\xi = 0, \qquad \eta(\xi) = 1, \qquad (2.1)$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \varphi Y) = -g(\varphi X, Y), \qquad (2.2)$$

$$\nabla_X \xi = \varphi AX, \qquad (\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi. \qquad (2.3)$$

Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi for any vector fields X, Y, Z on M are respectively given by

$$R(X, Y)Z = \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(\varphi Y, Z)\varphi X \qquad (2.4)$$

$$-g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z] + g(AY, Z)AX - g(AX, Z)AY,$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi], \qquad (2.5)$$

where R denotes the Riemannian curvature tensor on M.
Relation (2.4) implies that the structure Jacobi operator l is given by:

$$lX = \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - \eta(AX)A\xi. \qquad (2.6)$$

For every point $P \ \epsilon \ M$, the tangent space $T_P M$ can be decomposed as following:

$$T_P M = span\{\xi\} \oplus \mathbb{D}$$

where $\mathbb{D} = \{X \ \epsilon \ T_P M : \eta(X) = 0\}$. Due to the above decomposition, the vector field

$A\xi$ can be written:

$$A\xi = \alpha\xi + \beta U, \tag{2.7}$$

where $\beta = |\varphi\nabla_\xi\xi|$ and $U = -\frac{1}{\beta}\varphi\nabla_\xi\xi \ \epsilon \ ker(\eta)$, provided that $\beta \neq 0$.

3 Some previous results

In the rest of this paper, we use the notion $M_2(c)$, $c \neq 0$, to denote $\mathbb{C}P^2$ or $\mathbb{C}H^2$.

Let M be a non-Hopf hypersurface in $M_2(c)$. Then the following relations holds on every three-dimensional real hypersurface in $M_2(c)$.

Lemma 3.1 *Let M be a real hypersurface in $M_2(c)$. Then the following relations hold on M:*

$$AU = \gamma U + \delta\varphi U + \beta\xi, \qquad A\varphi U = \delta U + \mu\varphi U, \tag{3.1}$$

$$\nabla_U\xi = -\delta U + \gamma\varphi U, \qquad \nabla_{\varphi U}\xi = -\mu U + \delta\varphi U, \qquad \nabla_\xi\xi = \beta\varphi U, \tag{3.2}$$

$$\nabla_U U = \kappa_1\varphi U + \delta\xi, \qquad \nabla_{\varphi U}U = \kappa_2\varphi U + \mu\xi, \qquad \nabla_\xi U = \kappa_3\varphi U, \tag{3.3}$$

$$\nabla_U\varphi U = -\kappa_1 U - \gamma\xi, \ \nabla_{\varphi U}\varphi U = -\kappa_2 U - \delta\xi, \ \nabla_\xi\varphi U = -\kappa_3 U - \beta\xi, \tag{3.4}$$

where $\gamma, \delta, \mu, \kappa_1, \kappa_2, \kappa_3$ are smooth functions on M.

Proof: Let $\{U, \varphi U, \xi\}$ be an orthonormal basis of M. Then we have:

$$AU = \gamma U + \delta\varphi U + \beta\xi \qquad A\varphi U = \delta U + \mu\varphi U,$$

where γ, δ, μ are smooth functions, since $g(AU, \xi) = g(U, A\xi) = \beta$ and $g(A\varphi U, \xi) = g(\varphi U, A\xi) = 0$.

The first relation of (2.3), because of (2.6) and (3.1), for $X = U$, $X = \varphi U$ and $X = \xi$ implies (3.2).

From the well known relation: $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ for $X, Y, Z \ \epsilon \ \{\xi, U, \varphi U\}$ we obtain (3.3) and (3.4), where κ_1, κ_2 and κ_3 are smooth functions. $\qquad\square$

In [7], T.A.Ivey and P.J.Ryan proved the non-existence of real hypersurfaces in $M_2(c)$, whose structure Jacobi operator vanishes. In our context, we give a different proof of their Proposition 8 (non-Hopf case) and Lemma 9.

Proposition 3.2 *There does not exist real non-flat hypersurface in $M_2(c)$, whose structure Jacobi operator vanishes.*

Proof: Let M be a non-Hopf real hypersurface in $M_2(c)$, so the vector field $A\xi$ can be written $A\xi = \alpha\xi + \beta U$ (i.e. $\alpha\beta \neq 0$).

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Let $\{U, \varphi U, \xi\}$ denote an orthonormal basis of M. Since the structure Jacobi operator of M vanishes, from relation (2.6) for $X = U$ and $X = \varphi U$, we obtain: $AU = (\frac{\beta^2}{\alpha} - \frac{c}{4\alpha})U + \beta\xi$ and $A\varphi U = -\frac{c}{4\alpha}\varphi U$. Conversely, if we have a real hypersurface, whose shape operator satisfies the last relations then $l = 0$. Relations (3.2), (3.3) and (3.4) because of the latter become respectively:

$$\nabla_U \xi = (\frac{\beta^2}{\alpha} - \frac{c}{4\alpha})\varphi U, \qquad \nabla_{\varphi U} \xi = \frac{c}{4\alpha}U, \qquad \nabla_\xi \xi = \beta \varphi U, \tag{3.5}$$

$$\nabla_U U = \kappa_1 \varphi U, \qquad \nabla_{\varphi U} U = \kappa_2 \varphi U - \frac{c}{4\alpha}\xi, \qquad \nabla_\xi U = \kappa_3 \varphi U, \tag{3.6}$$

$$\nabla_U \varphi U = -\kappa_1 U - (\frac{\beta^2}{\alpha} - \frac{c}{4\alpha})\xi, \quad \nabla_{\varphi U}\varphi U = -\kappa_2 U, \quad \nabla_\xi \varphi U = -\kappa_3 U - \beta\xi, \tag{3.7}$$

where $\kappa_1, \kappa_2, \kappa_3$ are smooth functions on M.

On M the Codazzi equation for $X, Y \in \{U, \varphi U\, \xi\}$, because of (3.5), (3.6) and (3.7) yields:

$$U\beta = \beta\kappa_2(\frac{4\beta^2}{c} + 1), \tag{3.8}$$

$$\frac{\beta^2\kappa_3}{\alpha} = \beta\kappa_1 + \frac{c}{4\alpha}(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}), \tag{3.9}$$

$$U\alpha = \xi\beta = \frac{4\alpha\beta^2\kappa_2}{c}, \tag{3.10}$$

$$\xi\alpha = \frac{4\alpha^2\beta\kappa_2}{c}, \tag{3.11}$$

$$(\varphi U)\alpha = \beta(\alpha + \kappa_3 + \frac{3c}{4\alpha}), \tag{3.12}$$

$$(\varphi U)\beta = \beta^2 + \beta\kappa_1 + \frac{c}{2\alpha}(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}), \tag{3.13}$$

$$(\varphi U)(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}) = \beta(\frac{\beta^2}{\alpha} + \frac{\beta\kappa_1}{\alpha} - \frac{3c}{4\alpha}). \tag{3.14}$$

The Riemannian curvature on M satisfies (2.4) and on the other hand is given by the relation $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. The combination of these two relations implies:

$$U\kappa_3 - \xi\kappa_1 = \kappa_2(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} - \kappa_3), \tag{3.15}$$

$$(\varphi U)\kappa_3 - \xi\kappa_2 = \kappa_1(\kappa_3 + \frac{c}{4\alpha}) + \beta(\kappa_3 - \frac{c}{2\alpha}). \tag{3.16}$$

Relation (3.14), because of (3.9), (3.12) and (3.13), yields:

$$\kappa_3 = -4\alpha, \tag{3.17}$$

and so relation (3.9) becomes:

$$\beta\kappa_1 = \frac{c}{4\alpha}\left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\right) - 4\beta^2. \tag{3.18}$$

Differentiating the relations (3.17) and (3.18) with respect to U and ξ respectively and substituting in (3.15) and due to (3.10), (3.11) and (3.17) we obtain:

$$\kappa_2(c - 2\beta^2 - 4\alpha^2) = 0. \tag{3.19}$$

Owing to (3.19), we consider M_1 the open subset of points $P \in M$, where $\kappa_2 \neq 0$ in a neighborhood of every P. Due to (3.19) we obtain: $2\beta^2 + 4\alpha^2 = c$ on M_1. Differentiation of the last relation along ξ and taking into account (3.10), (3.11) and $2\beta^2 + 4\alpha^2 = c$ yields: $c = 0$, which is a contradiction. Therefore, M_1 is empty. Thus, $\kappa_2 = 0$ on M and relations (3.8), (3.10) and (3.11) become:

$$U\alpha = U\beta = \xi\alpha = \xi\beta = 0.$$

Using the above relations we obtain:

$$[U, \xi]\alpha = U\xi\alpha - \xi U\alpha = 0,$$

$$[U, \xi]\alpha = (\nabla_U\xi - \nabla_\xi U)\alpha = \frac{1}{4\alpha}(4\beta^2 + 16\alpha^2 - c)(\varphi U)\alpha.$$

Combining the last two relations we have:

$$(4\beta^2 + 16\alpha^2 - c)(\varphi U)\alpha = 0. \tag{3.20}$$

Let M_2 be the set of points $P \in M$, for which there exists a neighborhood of every P such that $(\varphi U)\alpha \neq 0$. So in M_2 from (3.20) we have: $16\alpha^2 + 4\beta^2 = c$. Differentiating the last relation with respect to φU and taking into account (3.12), (3.13), (3.17), (3.18) and $16\alpha^2 + 4\beta^2 = c$, we obtain: $4\alpha^2 + \beta^2 = 0$, which is impossible. So M_2 is empty. Hence, on M we have $(\varphi U)\alpha = 0$. Then, relations (3.12), (3.17) and (3.18) imply: $c = 4\alpha^2$ and $\beta\kappa_1 = \alpha^2 - 5\beta^2$. On the other hand from relation (3.16), because of (3.17) we obtain: $\kappa_1 = -2\beta$. Substitution of κ_1 in $\beta\kappa_1 = \alpha^2 - 5\beta^2$ yields: $3\beta^2 = \alpha^2$. Taking the covariant derivative along φU of $3\beta^2 = \alpha^2$, because of (3.13), we conclude: $\beta = 0$, which is a contradiction.

Suppose that $A\xi = \beta\xi$ (i.e. $\alpha = 0$ and $\beta \neq 0$). Since the structure Jacobi operator of M vanishes, from relation (2.6) for $X = \varphi U$, we obtain: $c = 0$, which is impossilbe.

Hence, there do not exist non-Hopf hypersurfaces with $l = 0$. Using this and the Hopf case ([7]), we complete the proof of the present Proposition. $\qquad\square$

4 Auxiliary Relations

If M is a real hypersurface in $M_2(c)$, we consider the open subset $\mathcal{N}$ of M such that:

$$\mathcal{N} = \{P \ \epsilon \ M : \ \beta \neq 0, \ \text{ in neighborhood of P}\}.$$

Furthermore, we consider $\mathcal{V}$, Ω open subsets of $\mathcal{N}$ such that:

$$\mathcal{V} = \{P \ \epsilon \ \mathcal{N} : \alpha = 0, \ in \ a \ neighborhood \ of \ P\},$$

$$\Omega = \{P \ \epsilon \ \mathcal{N} : \alpha \neq 0, \ in \ a \ neighborhood \ of \ P\},$$

where $\mathcal{V} \cup \Omega$ is open and dense in the closure of $\mathcal{N}$.

Lemma 4.1 *Let M be a real hypersurface in $M_2(c)$, equipped with pseudo-parallel structure Jacobi operator. Then $\mathcal{V}$ is empty.*

Proof: Let $\{U, \varphi U, \xi\}$ be a local orthonormal basis on $\mathcal{V}$. The relation (2.7) takes the form $A\xi = \beta U$ and we consider:

$$AU = \gamma' U + \delta' \varphi U + \beta \xi, \qquad A\varphi U = \delta' U + \mu' \varphi U, \tag{4.1}$$

since $g(AU, \xi) = g(U, A\xi) = \beta$, $g(A\varphi U, \xi) = g(\varphi U, A\xi) = 0$ and γ', δ', μ' are smooth functions.

From (2.6) for $X = U$ and $X = \varphi U$, taking into account (4.1), we obtain:

$$l\varphi U = \frac{c}{4}\varphi U \qquad lU = (\frac{c}{4} - \beta^2)U. \tag{4.2}$$

Relation (1.1) for $X = U$, $Y = \xi$ and $Z = \varphi U$, because of (2.4), (4.1) and (4.2) yields: $\delta' = 0$, since $\beta \neq 0$.

Furthermore, relation (1.1) for $X = U$ and $Y = Z = \varphi U$, owing to (2.4), (4.1), (4.2) and $\delta' = 0$ implies:

$$\mu' = 0 \qquad c = L, \tag{4.3}$$

and for $X = \xi$ and $Y = Z = \varphi U$, because of (4.3), gives: $c = 0$, which is a contradiction. Therefore, $\mathcal{V}$ is empty. $\qquad\qquad\square$

In what follows we work on Ω, where $\alpha \neq 0$ and $\beta \neq 0$.

By using (2.6) and relations (3.1) we obtain:

$$lU = (\frac{c}{4} + \alpha\gamma - \beta^2)U + \alpha\delta\varphi U \qquad l\varphi U = \alpha\delta U + (\alpha\mu + \frac{c}{4})\varphi U \tag{4.4}$$

The relation (1.1) because of (2.4), (3.1) and (4.4), implies:

$$\delta \ = \ 0, \ \text{for } X = U, \ Y = \xi \ \text{ and } \ Z{=}\varphi U, \tag{4.5}$$

and additional due to (4.5) yields:

$$\mu(\alpha\mu + \frac{c}{4}) \quad = \quad 0, \quad \text{for } X = U, \ Y = \varphi U \ \text{ and } \ Z = \xi. \tag{4.6}$$

Owing to (4.6), we consider Ω_1 the open subset of Ω, such that:

$$\Omega_1 = \{P \ \epsilon \ \Omega : \mu \neq -\frac{c}{4\alpha}, \ \text{ in } \ a \ \text{ neighborhood } \ of \ P\}.$$

Therefore, in Ω_1 from (4.6) we have: $\mu = 0$.

Lemma 4.2 *Let M be a real hypersurface in $M_2(c)$, equipped with pseudo-parallel structure Jacobi operator. Then Ω_1 is empty.*

Proof: In Ω_1, relation (1.1) for $X = U, Y = \varphi U$ and $Z = U$, because of (2.4), (3.1), (4.4) and (4.5) yields:

$$(\beta^2 - \alpha\gamma)(c - L) = 0. \tag{4.7}$$

Due to (4.7), we consider the open subset Ω_{11} of Ω_1, such that:

$$\Omega_{11} = \{P \ \epsilon \ \Omega_1 : c \neq L, \ \text{ in } \ a \ \text{ neighborhood } \ of \ P\}.$$

So in Ω_{11}, we obtain: $\gamma = \frac{\beta^2}{\alpha}$.

In Ω_{11}, the relation (2.5), because of Lemma 3.1 and (4.5), yields:

$$\frac{\beta^2 \kappa_3}{\alpha} \quad = \quad \beta\kappa_1 + \frac{c}{4}, \quad \text{for } X = U \ \text{ and } \ Y = \xi \tag{4.8}$$

$$(\varphi U)\alpha \quad = \quad \beta(\alpha + \kappa_3), \quad \text{for } X = \varphi U \ \text{ and } \ Y = \xi \tag{4.9}$$

$$(\varphi U)\beta \quad = \quad \beta^2 + \beta\kappa_1 + \frac{c}{2}, \quad \text{for } X = \varphi U \ \text{ and } \ Y = \xi \tag{4.10}$$

$$(\varphi U)\frac{\beta^2}{\alpha} \quad = \quad \frac{\beta^2}{\alpha}(\kappa_1 + \beta), \quad \text{for } X = U \ \text{ and } \ Y = \varphi U. \tag{4.11}$$

Substituting in (4.11) the relations (4.9), (4.10) and taking into account (4.8) we obtain: $\frac{3c\beta}{4\alpha} = 0$, which is a contradiction. Therefore, Ω_{11} is empty and $L = c$ in Ω_1.

In Ω_1, relation (1.1) for $X = \xi$ and $Y = Z = \varphi U$, because of (2.4), (3.1) and (4.4) implies: $c = 0$, which is impossible. Therefore, Ω_1 is empty. $\square$

From Lemma 4.1, we conclude that $\mu = -\frac{c}{4\alpha}$ in Ω.

Lemma 4.3 *Let M be a real hypersurface in $M_2(c)$, equipped with pseudo-parallel structure Jacobi operator. Then Ω is empty.*

Proof: In Ω, relation (1.1) for $X = \varphi U, Y = \xi$ and $Z = U$, due to (2.4), (3.1), (4.4) and (4.5) yields: $\gamma = \frac{\beta^2}{\alpha} - \frac{c}{4\alpha}$. Owing to $\mu = -\frac{c}{4\alpha}$ and $\gamma = \frac{\beta^2}{\alpha} - \frac{c}{4\alpha}$ and (4.5), relation (4.4) implies: $lU = l\varphi U = 0$ and since $l\xi = 0$, we obtain that the structure Jacobi operator

vanishes in Ω. Due to Proposition 3.2, we conclude that Ω is empty. $\square$

From Lemmas 4.1 and 4.3, we conclude that $\mathcal{N}$ is empty and we lead to the following result:

Proposition 4.4 *Every real hypersurface in $M_2(c)$, equipped with pseudo-parallel structure Jacobi operator, is a Hopf hypersurface.*

5   Proof of Main Theorem

Since M is a Hopf hypersurface, due to Theorem 2.1 ([11]) we have that α is a constant. We consider a unit vector field $e \in \mathbb{D}$, such that $Ae = \lambda e$, then $A\varphi e = \nu\varphi e$ at some point $P \in M$, where $\{e, \varphi e, \xi\}$ is a local orthonormal basis. Then the following relation holds on M, (Corollary 2.3 [11]):

$$\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}. \tag{5.1}$$

The relation (2.6) implies:

$$le = (\frac{c}{4} + \alpha\lambda)e \quad \text{and} \quad l\varphi e = (\frac{c}{4} + \alpha\nu)\varphi e. \tag{5.2}$$

Relation (1.1) for $X = e$ and $Y = Z = \varphi e$, because of (2.4) and (5.2) yields:

$$\alpha(c + \lambda\nu - L)(\nu - \lambda) = 0. \tag{5.3}$$

Relation (1.1) for $X = Z = e$, $Y = \xi$ and for $X = Z = \varphi e$, $Y = \xi$, because of (2.4) and (5.2) implies respectively:

$$(\frac{c}{4} + \alpha\lambda)(L - \alpha\lambda - \frac{c}{4}) = 0, \tag{5.4}$$

$$(\frac{c}{4} + \alpha\nu)(L - \alpha\nu - \frac{c}{4}) = 0. \tag{5.5}$$

Because of (5.3), we consider $\mathcal{M}_1$ the open subset of M, such that:

$$\mathcal{M}_1 = \{P \in M : \alpha(\nu - \lambda) \neq 0 \ \ in \ \ a \ \ neighborhood \ \ of \ \ P\}.$$

So in $\mathcal{M}_1$, we have: $L = c + \lambda\nu$.

Proposition 5.1 *Let M be a real Hopf hypersurface in $M_2(c)$, equipped with pseudo-parallel structure Jacobi operator. Then $\mathcal{M}_1$ is empty.*

Proof: Because of (5.4), we consider $\mathcal{M}_{11}$ the open subset of $\mathcal{M}_1$, such that:

$$\mathcal{M}_{11} = \{P \in \mathcal{M}_1 : \ L \neq \alpha\lambda + \frac{c}{4}, \ in \ a \ neighborhood \ of \ P\}.$$

In $\mathcal{M}_{11}$ relations (5.4) and (5.5) imply: $\lambda = -\frac{c}{4\alpha}$ and $L = \alpha\nu + \frac{c}{4}$, respectively since $\lambda \neq \nu$. Using the last two relations and because of $L = c + \lambda\nu$ and (5.1), we obtain:

$$\lambda = \frac{4\alpha}{7}, \qquad \nu = -4\alpha, \qquad c = -\frac{16\alpha^2}{7}. \tag{5.6}$$

Because of (5.6), we have $c < 0$ and three distinct constant eigenvalues. So the only case is real hypersurface of type B in $\mathbb{C}H^2$. Substitution of the eigenvalues of type B real hyper-surfaces (see [1]) in (5.6), leads to a contradiction. So $\mathcal{M}_{11} = \emptyset$. Consequently, in $\mathcal{M}_1$ the relation $L = \alpha\lambda + \frac{c}{4}$ holds and because of (5.5), we lead to: $\nu = -\frac{c}{4\alpha}$, since $\lambda \neq \nu$. Following the same method as above, we obtain a contradiction and this completes the proof of the Proposition. $\qquad\square$

Thus from Proposition 5.1, we conclude that $\alpha(\nu - \lambda) = 0$ at any point $P \epsilon M$. Thus locally either $\alpha = 0$ or $\nu = \lambda$.

If $\alpha = 0$ in case of $\mathbb{C}P^2$, M is locally congruent to a tube of radius $r = \frac{\pi}{4}$ over a holomorphic curve in $\mathbb{C}P^2$, if $\lambda \neq \nu$ or to a geodesic hypersphere of radius $r = \frac{\pi}{4}$, if $\lambda = \nu$, (see [3]), and in case of $\mathbb{C}H^2$, M is a Hopf hypersurface with $A\xi = 0$.

If $\alpha \neq 0$, we have: $\lambda = \nu$. Then $Ae = \lambda e$ and $A\varphi e = \lambda\varphi e$, therefore we obtain:

$$(A\varphi - \varphi A)X = 0, \ \ \forall \ X \ \epsilon \ TM.$$

From the above relation Theorem 1.1 holds and this completes the proof of Main Theorem.

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